PRODUCTION CAPACITY FUNCTION IN MANUFACTURING

by

Bogdan Vorotilin

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Abstract

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by Bogdan Vorotilin

KSE Program Director: Tom Coupé

The focus of the thesis is upon the concept of production capacity and the factors that determine it. The paper aims at formulation of the production capacity function for manufacturing that would include all relevant factors of production and its verification.

In the first chapter a brief literature review is presented. It comprises the relevant studies from the fields of economics and production management, and a number of bordering papers. However, the problem raised is still under researched.

In the second chapter the set of propositions is advanced. Those propositions are designed to capture the most essential features of manufacturing production. They are split into five blocks, each dealing with a major set of factors within the production. Those are considered to be process organization, labour, and workers' professional skills, operating time, and reject rate of production.

In the third chapter, the framework itself is introduced. Then it is verified analytically as to whether it satisfies each of the propositions from the set advanced. Upon verification, the framework appears to satisfy the entire set of propositions. However, some of them are satisfied only partially. Yet this drawback is minor since it involves relatively short intervals of the admitted regions of the variables.

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GLOSSARY

Assembling is a production stage during which subunits are assembled into a composite product

Attendant processes support successful performance of primary and auxiliary processes, e.g. transportation, storage, etc.

Auxiliary processes are the processes that provide a continuous passing of the primary processes, e.g. repair, etc.

Blank production includes blank manufacturing processes

Complex production process is a set of simple production processes

Manufacturing consists in the transformation of blanks into complete products

Operational cycle determines a set of technological processes executed at a workbench

Primary production process is a production process during which article is manufactured

Process organization includes: production structure; spatial arrangement; time profile; job design

Production process is a set of operations of workers and implements executed at the production of particular article on the basis of particular technology

Production cycle is a set of primary, auxiliary and attendant processes arranged in time that are necessary for the production of particular product

Simple production process is a set of successive operations executed upon a single object of labor

Technological processes are purposive actions aimed at transformation of objects of labor

Technology is a succession of transformations of objects of labor into finished article by the means of well-directed use of natural processes and mechanical effect

INTRODUCTION

With the rapid advance of technology in manufacturing as well as in a broad economy, production activities become extremely complex. Not only do many corporations produce different products but they may and do produce them differently at different locations. For example ArcelorMittal Kryvyi Rih uses both blast and open-hearth furnaces at its steel mill in Kryvyi Rih, Ukraine. Though both production processes share the same technology they are totally different with respect to the equipment sets and processes employed. This feature is representative for the majority of corporations and even smaller firms.

This complexity substantially diminishes management's capability to detect sources of inefficiency in production not to mention sales. In regular practice managers keep the records of production and identify events that cause drops in output to control the production process. Analytical tools they use are for the most part qualitative. Upon this qualitative analysis managers determine the actions to target factors that have negative impact upon production.

In many simple situations qualitative analysis may suffice but at a distressed firm or a firm with steady negative or extremely fluctuating output growth rates, an improved analytical framework for the detection of efficiency leaks and adequate corrective measures is needed. The main problem with qualitative analysis is that it doesn't employ marginal effects and magnitudes. Using qualitative tools, the manager is unable to determine the magnitude of inefficiency which corresponds to a particular factor he wants to target. He is also unable to predict the magnitude of improvement from his corrective measures. Obviously there is a room for improvement.

In the present paper I will attempt to formulate the framework for the quantitative analysis of production capacity of enterprises. The study will focus on primary production processes of enterprises abstracting from the issues related to auxiliary and attendant processes that will stay out of scope of this research. The problems of taxonomy will also be kept out of the paper. Upon the completion the framework will include either directly or optionally engineering variables, elements of job design, materials allocation, learning options, motivation, management, etc. The framework will incorporate relevant findings in the fields of economics and operation management.

Since the framework aims at a practical application rather than learning purposes simplifications will be reduced to minimum. In practice it means that the number of relevant variables may well reach into dozens and links between them may happen to be nonlinear. This is done in order to achieve the maximum precision in calculation which is crucial for the purposes of planning and control.

Although the framework incorporates an extensive number of variables data on most of which is not publicly disclosed by enterprises and thus is unavailable for empirical study at the moment, it is not a sufficient barrier to prevent the study of this issue. The problem with data may well be resolved in the future either through additional government requirements for the disclosure of complementary statistical information on firms' activities or through the natural process of publicity expansion stimulated by capital markets as it happened with financial data. Given the data constraints the empirical issues of research will be discussed in subsequent papers.

The crucial point for the comprehension of the paper is the absence of confusion regarding terminology and concepts. At first, I wish to prevent the confusion of concepts of production capacity and production function as it proves to be the worst obstacle. Production function concept is widely used in economics and is related to production capacity concept used in process management. To elucidate the difference I provide the definitions of the both concepts.

Production capacity defines a volume of products that can be generated by a production plant or enterprise in a given period by using current resources[1]

Production function describes a mapping from quantities of inputs to quantities of an output as generated by a production process[2]

The main point of difference between production capacity and economics production function is that the first applies to individual enterprise while the second does not.

The heavy use of production engineering terminology will be resorted to at some points in the paper. Therefore, the vocabulary of specific terms will be provided at the end of the paper to ease the comprehension.

Chapter 1

LITERATURE REVIEW

Issues of production management had not received much of the scrupulous attention of economists who chose to concentrate upon the concept of aggregate production function as an easier one for comprehension and manipulation. The choice was dictated by three factors, namely ultimate goals, data availability and computational constraints. Economists aimed at explaining macro phenomena and thus needed some aggregate measures of labor, capital, materials, etc. Since those factors were heterogeneous, aggregation was made using costs data derived from accounting reports. First steps at construction of aggregate production function were taken in the late 19th century by H. von Thunen[3] and P. Wiksteed[4]. However, it is not until the paper of W. Cobb and P. Douglas in 1920s that economists started to employ the concept of production function in their research. Mathematical form of production function derived by Cobb and Douglas displayed following important properties:

- direct functional relation between inputs and outputs of production;
- limit upon the number of input variables to aggregated capital and labor and abstraction from all other constituents of production process;
- focus upon marginal effects of increasing inputs;
- econometric estimation of production function that represents all important characteristics of production other than labor and capital in the form of disturbance or construction of meta-production functions which are specific for every technology.

Great improvement of the production function concept came with the work of K. Arrow and H. Chenery[5] who managed to derive the functional form which allowed the elasticity of substitution between factors of production to vary. Later improvements came with the introduction of translog[6] and Diewert[7] functions. Both allowed more precise estimation through the mimicry of the second-order Taylor polynomial of the hypothetical production function with slight differences.

Although the concept of production function had gradually become very advanced its applicability to existing production and business practices remained insignificant. This fact results from the design of the modern production functions which implies simplification, econometric estimation, and great degree of approximation. Those design features are beneficial for the macro estimation but generate large distortions if applied at a micro level. Various economists approached this issue in the recent sixty years. The first attempt was made by H. Chenery in the 1940s who introduced engineering analysis into the production function.[8] He argued that combination of engineering and economic methodology may greatly improve the performance and applicability of economists' models. In the following years a number of economists considered the ways engineering data can be used in economics of production. M. Kurz and A. Manne in 1963 estimated the production function of metal machining using the engineering data. [9] Their study ignited vivid discussion in the ranks of economists. E. Furubotn[10] and L. Lawe[11] questioned the validity of Kurz and Manne's censoring rule while approving the basic principles of the approach. In the early 1970s the issue spurred the attention of economists again. D. Levhari made an attempt to introduce the elements of operation management into the production theory.[12] He developed the model which described the output as a function of the random process of machinery failures in a fully automated production and repair services. Thus, he actually combined primary and auxiliary processes into one equation. The benefits of engineering data usage were next year outlined by R. Anderson who argued in his article that engineering information may enable economists to detect structural changes in production function equations way before relevant information appears in the accounting books. [13] In the 1980s another attempt was made to estimate the production function with engineering data. T. Sav drawing heavily upon the early paper of H. Chenery estimated the production function for solar process application.[14] However, those multiple attempts fell short of firmly introducing process analysis and engineering data utilization into economic analysis despite the plentiful benefits it could possibly convey to the production theory.

In contrast, operation management initialized by F. Taylor in the early 1910s in his "Principles of scientific management" approached rapidly towards the introduction of the production capacity function.[15] At the current stage, production management possesses extensive taxonomy of production related phenomena such as production and process classification, etc.[16] This taxonomy and slightly modified definition set will form the basis of my research. Multiple aspects of production are already thoroughly investigated within the operation management framework. Those include technology issues, learning, training, motivation of the personnel, job design, process layout, etc.[17] Since the focus of operation management shifted heavily towards the time and material requirements of specific productions with less emphasis upon cost figures, the results obtained were more stable and significant at the micro level than those of economists who made primary use of accounting data and targeted more general entities. However, findings of engineers are heavily fragmented and at some points skewed towards the qualitative conclusion which inhibits integral comprehension of the production process.

In my research I will take advantage of the taxonomy developed by engineers and use their sets of inputs with slight modifications. My attention will be focused primarily upon the aspects of the production which are under direct control of management to assure the highest compatibility with application.

Chapter 2

PROPOSITION SET

The basic concept that underlies the manufacturing practices in every industry is technology. When technology is mentioned, it implies that there exists a succession of transformations of objects of labor into finished article by the means of well-directed use of natural processes and mechanical effect. In engineering design technology takes the form of the flowchart, which describes the sequence of steps necessary to transform material inputs into the unit of output. Take micro chip industry as an example. The underlying technology of chip production is similar for either high or low performance processors. In either case technological steps involved are divided into wafer fabrication, die preparation, packaging and testing.[18] However, despite the technology sharing, manufacturing practices differ for different chip architectures. This fact is captured by the concept of production process. The latter describes a set of operations of workers and implements executed at the production of a particular article on the basis of a particular technology. From this definition we can extract three determinants of the production process, namely: operations set, technology, and implements. In case of micro chip industry differences in the production process of cores based on different architectures will take the form of different mask patterns used in the process of photolithography. The differences in the production processes for chip industry are so large that the new fab is required for every new architecture.

As the concepts of technology and production process are introduced, we approach the issue of the selection of the key factors that determine the capacity of the production unit. The major determinant of the production capacity is production process employed, which in turn defines technology, set of operations, and equipment set. At this point the first obstacle arises. The heart of the problem consists in the absence of the quantitative measure for the production process. It can be represented qualitatively through the flowchart but this is not an adequate solution in the situation when we have to come up with actual figures. However, a bypass to this problem does exist. Production process can be represented as a set of time parameters that enter into the production capacity function and are determined exogenously through the inner structure of production process and equipment measures, which will be defined later. The next most important factor in the determination of production capacity is process organization which includes production structure, spatial arrangement, time profile, and job design. This factor determines the time and space structure of the production process and is the third most significant source of constant productivity improvement after technology advance and equipment progress.

Among important factors we also list the number of workers, their professional skills and motivation, defective articles ratio generated in production, sick leaves ratio and various time losses which are grouped into categories. There is no need to talk in great detail about the importance of such factor as the number of workers since it is close to the concept of labor in producer theory. However, we have to focus on such factors as motivation, professional skills, etc., the impact of which upon the production is seriously underestimated by the economists in favor of capital and labor factors. To illustrate the importance of the above mentioned factors we have to turn to the experience of USSR, which clearly demonstrated the detrimental effect of degraded workers' motivation, which partially showed up in excess sick leaves and inflated time losses due to absenteeism and related causes.

At this point, a number of assumptions are introduced in order to simplify the exposition:

1. Only one type of workers is employed
2. Production is sequential
3. Only primary production processes are considered
4. Materials are abundant

Essentially the model of production capacity relies on a single identity between quantity of output and time:

$$Q(N = 1) = \frac{T}{t}$$

Where Q - quantity of output or vector of quantities in general case, measured in units

N – number of equipment sets defined by the production process

T – operating time

t – duration of the production cycle

The expression shows that in case of a single equipment set the number of units of output produced is equal to the number of repetitions of production cycle within the given operating time.

If we allow for multiple equipment sets to be simultaneously employed in production we will get Q as a function of additional parameter j that enters the expression in the form of an index and defines the ordering of equipment sets:

$$Q = T\sum_{j=1}^{N}\frac{1}{t_j}$$

However, equipment can be introduced into the expression in the form of a variable in the case of replication when the number of workers per equipment set, process organization, and all relevant parameters are kept constant. In this case the expression becomes:

$$Q = \frac{NT}{t}$$

Where N – number of equipment sets

In our further exposition we will take advantage of the simplicity of replication form representation and will assume the situation of identical relevant parameters for every equipment set employed.

The introduction of every new factor relevant for the production capacity model takes place through consecutive modifications of original identity. Factors are introduced in blocks. Therefore, the exposition will be structured accordingly. As a first step, the set of propositions is advanced that are argued to correctly describe the behavior of the production capacity function. Then each of the propositions is verified analytically. As a result, we obtain the production capacity function, which simultaneously satisfies the entire proposition set.

The proposition set is structured into blocks, which collect all related factors into one bundle to advance the ease of comprehension.

Block 1. Process organization and labor

This block comprises the propositions which define the impact of production process, labor and process organization upon the capacity of the production unit. Despite the fact that those factors cannot be measured directly we are able to overcome this difficulty by representing them as a set of time parameters estimated from the data collected from field experiments in production.

Proposition 1.1 *As long as the number of workers per equipment set doesn't meet the minimal requirement, production cycle is infinitely long.*

In simple language this proposition states that for every production process there is a minimal number of workers for the production to begin. Until at least this number of workers is employed the production capacity stays at zero. This number is different for every production process and is often determined in the flow chart. For the majority of production processes this number takes a finite positive value. In extreme cases it may either tend to zero like at fully automated assembly lines or get extremely large as it is the case for steel plants. To get the specific value for this parameter we have to consult the flowchart of the production process of interest.

Proposition 1.2 *Under efficient labor organization, duration of the production cycle tends to decrease as the number of workers increases but at a diminishing rate*

Efficient labor organization implies an existence of efficient frontier of production cycle duration function, which constitutes a set of points representing mathematical solutions of multi parameter production problem, that includes such parameters as spatial arrangement, job design, time profile, and production structure. Under efficient labor organization additional workers will never decrease the overall performance of the production unit. To prove this statement let's suppose that additional worker actually decreases the overall performance if he is integrated into the production process. Then at the efficient frontier this worker will be kept out of production process and thus marginal effect of his employment upon the overall performance will constitute zero. Therefore, at the efficient frontier marginal effect of additional employee upon the duration of production cycle is always equal to or less than zero. The absolute value of the marginal effect decreases with every additional employee since we keep the number of equipment sets constant. Thus the contribution of each successive employee to the efficiency of the production decreases in magnitude.

This proposition is analogous to the one used in producer theory.

Proposition 1.3 *As number of employees gets sufficiently great and inefficiencies of organization are eliminated, duration of the production cycle tends to the autonomous level determined by the production process and professional skills level of employees*

Diminishing magnitude of marginal effect of additional employee results in the convergence of the production cycle duration to a finite positive number. This level is reached when marginal effect of additional employee becomes close to zero. If we stay on the efficient frontier, then after the effect of additional labor is exhausted the duration of the production cycle is determined by other factors, namely professional skills level of workers employed and the nature of production process. This proposition insures that production capacity will never turn into infinity as long as the operating time and the number of equipment sets are finite.

Block 2. Professional skills

This block comprises the propositions that define the impact of the professional skills of employees upon the capacity of the production unit. Professional skills are exhaustively determined by experience, training, and ability. Experience and training themselves are split into direct and related ones. While direct experience measures the aggregate amount of time spent on execution of the particular set of operations, which a person currently executes in production of a specific item, related experience accounts for all production activity of individual that generates experience which can possibly be employed at his current workplace. The same logic applies to training.

Experience and training are relatively easy in terms of measurement and data collection. Both variables are expressed in time units. Information on them is contained in service records and CVs of employees.

The situation with ability is different. The mainstream approximates ability by IQ test score. However, this approach is biased in production, since majority of the production activities involve not only intellectual but also physical ability. While the first concept determines how fast a person adapts the direct knowledge obtained in the process of training to the production activity and arranges the workplace to facilitate the work, the second one captures the impact of constitution upon the performance of duties. In the model ability is expressed as a set of parameters whose values are specific to every production process.

Proposition 2.1 *Duration of the production cycle decreases as direct experience increases but at a diminishing rate.*

This proposition basically states that marginal effect of experience upon the production capacity is positive but its magnitude decreases as higher experience levels are attained. This insures that production capacity can't be extended indefinitely by higher levels of experience.

Proposition 2.2 *Production of the first unit takes a finite amount of time.*

This proposition insures that a newly trained employee is able to start producing. If this condition is not satisfied, it will take infinitely long for an employee without previous experience to produce his first item of product.

Proposition 2.3 *As direct experience gets sufficiently great, duration of the production cycle tends to the value determined by the production process, process organization efficiency and the number of workers.*

In the real world no matter how high worker's experience level is, he is still constrained by other factors of production, namely production process, process organization efficiency and the number of workers simultaneously employed in production. This results in the fact that the production cycle duration converges to a positive value instead of zero. This proposition insures that convergence is reached.

Proposition 2.4 *Production cycle is infinitely long if workers don't possess necessary direct training in the amount that enables them to produce.*

Before a worker starts to execute a set of operations at production, he has to be trained to get the knowledge of the procedures and algorithm of operation execution. This is true for all even basic production processes. However, the amount of minimum training is different for various production processes. Ones with a high degree of complexity require greater amount of training than those less complex. Let's take micro processor production. In this industry it takes years of training for a technician before he is employed, whereas in agriculture seasonal workers are trained within one hour.

Proposition 2.5 *Duration of the production cycle decreases as amount of direct training increases but at a diminishing rate.*

This proposition basically states that marginal effect of training upon the production capacity is positive but its magnitude decreases as higher training levels are attained. This insures that production capacity can't be extended indefinitely by higher levels of training.

Proposition 2.6 *As direct training increases to a sufficiently high level, production cycle duration tends to the value determined by production process, process organization efficiency, the number of workers, their direct and related experience and training, and ability parameters.*

As employee is constrained by other factors of production besides direct training the value of the production cycle duration stays positive even after the effect of training exhausts itself. Therefore, the impact of training on the production capacity is limited.

Proposition 2.7 *Low quality direct training doesn't affect the duration of the production cycle.*

The quality of training constitutes a composite term of a person's effort endowed into learning and the expertise level of the trainer. If both parameters stay sufficiently low, the overall effect of training upon the expertise of an employee becomes negligible.

Proposition 2.8 *Direct training substitutes the contribution of related training and experience in the process of production.*

This means that the use of the training and experience obtained in related field diminishes with the higher levels of direct training. As a worker gets more useful information about the technological processes, procedures and algorithm of operations, his need to resort to related experience and training to facilitate his work and improve performance decreases.

Proposition 2.9 *Duration of the production cycle decreases as the amount of related training and experience increases but at a diminishing rate.*

The reasoning is analogous to the one behind Proposition 2.6

Proposition 2.10 *As related training and experience increase to a sufficiently high level, the production cycle duration tends to the value determined by production process, process organization efficiency, the number of workers, their direct training, and ability parameters.*

The reasoning is analogous to the one behind Proposition 2.7

Block 3. Sick leaves

Every employee gets sick from time to time and goes on sick leave. Although performance losses due to this factor are much smaller than due to the inefficiencies of process organization or inadequate professional skills of workers, they may be sufficiently high in absolute terms when it comes to the large enterprises. This block comprises the propositions that define the impact of this phenomenon upon the capacity of the production unit.

Proposition 3.1 *Production cycle duration within a designated period is an expected value of the range of durations generated by varying number of employees with probabilities defined by the frequency of sick leaves occurrence.*

As sick leaves are random they cannot be integrated into the schedule. This creates a situation when the number of employees simultaneously performing their duties becomes a random variable. That, in turn, randomizes the production cycle duration. Therefore, the mean value of the latter, which is to be used in the calculation of production capacity, has to constitute its expected value.

Proposition 3.2 *The presence of sick leaves ratio decreases the duration of operating time.*

According to Proposition 1.1, there exists the minimal number of workers for the production to start. In the times of pandemics we can expect the number of workers staying on sick leaves to be very large that can cause a complete halt of the production due to employee shortages. In that situation the expected value of the production cycle duration can not be computed unless we transfer the impact of halts upon the duration of the operating time.

Block 4. Operating time

Production as any process may be brought to a standstill by the impact of internal and external factors. However, it is also possible to partially compensate this loss by scheduling overtime hours in cases it is possible. This block of propositions defines the impact of those factors upon the capacity of the production unit.

Proposition 4.1 *Expected duration of the operating time is less than or equal to scheduled one.*

Every time period working schedule for a particular production unit is drawn. However, we can expect it to be under fulfilled due to such factors as machinery failures, inspections, accidents, etc. that happen regularly and may halt the production entirely. That's why we employ the concept of expected operating time to account for possible time losses.

Block 5. Reject rate

Not every item of product manufactured meets the technical requirements. In severe cases such articles are either remade or scrapped upon detection. In many production processes reject rate stays well above five percent. Let's take display adapters as an example. Data shows that even after testing the failure rate of some vendors for particular models may hit a ten percent level.[19] This substantially reduces the capacity of the production unit. Rejects are generated by a number of factors that can be split into two groups, namely human and machinery related. This block of propositions will define the nature of interactions of the factors that ultimately determine the reject rate of production.

Proposition 5.1 *Factors that determine the reject rate of production are independent, collectively exhaustive, and nonexclusive.*

Nonexclusiveness of the impact of different factors may be demonstrated upon the example. Let's take equipment setup quality and wear factor of machinery. The faults of setup obviously do not prevent rejects to be generated by degraded machinery performance due to severe wear and tear and vice versa. These two as well as other factors can act simultaneously. Impact of generic features of labor and machinery, professional skills of employees, wear factor of equipment and accuracy of equipment setup collectively exhausts the probability of reject occurrence. No other factor can generate a reject. To demonstrate the independence of factors, let's analyze generic labor features and professional skills. Generic labor features comprise such elements as varying motivation, health status, and generic imperfection of human concentration. They are clearly not influenced by professional skills status and act independently of it. Vice versa is also true.

Proposition 5.2 *Zero equipment wear factor, sufficiently high professional skills of employees and correct equipment setup reduce reject rate to the level determined by generic features of machinery and labor.*

Even in case of new equipment, highly professional team and perfect equipment setup coincidence rejects will not be eliminated. This results from the presence of generic imperfections in humans and machinery.

Proposition 5.3 *Each separate factor can effectively reduce capacity to zero as its intensity gets sufficiently high.*

In real life factors are not constrained in the amount of damage they can cause. The only constraint constitutes the set of preventive measures taken in production to minimize the loss.

Proposition 5.4 *At zero direct experience level reject rate increases to a hundred percent.*

Employee with no direct production experience will start his career by producing at the maximum reject rate.

Proposition 5.5 *Reject rate decreases as amount of direct experience increases but at a diminishing rate.*

This proposition basically states that marginal effect of experience upon the failure rate is negative but its magnitude decreases as higher experience levels are attained.

Proposition 5.6 *Reject rate decreases as amount of direct training increases but at a diminishing rate.*

The reasoning is analogous to the one behind Proposition 5.5

Proposition 5.7 *As direct training increases to a sufficiently high level reject rate stays positive.*

This proposition states that reject elimination cannot be achieved by a mere increase in the amount of training.

Proposition 5.8 *Reject rate increases with an increase in equipment wear factor at an increasing rate.*

As components of machinery degrade an isochronously we can expect this process to start accelerating as wear mounts affecting the performance of new components. That in turn makes reject rate to grow nonlinearly as wear fades in.

C h a p t e r 3

FRAMEWORK FORMULATION AND VERIFICATION

In the previous chapter a set of propositions was advanced which as we argued has to be satisfied by the framework to be considered consistent with production practice. In the current chapter the framework itself will be formulated. Then analytical verification of every single proposition will be performed.

Consider the following functional form and a set of constraints upon the parameters it includes:

Model

$$
\begin{cases}
Q=(1-da)N\cdot\dfrac{E(T)}{E(t)}\\
da=1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)\\
f=\left(1-(\phi+\delta)\right)\\
\delta=\dfrac{\eta V}{V+c}\\
V=qtr-tr^{*}\\
E(T)=\left[1-\sum_{l=\tilde{L}}^{L}P_L(l)\right](1-TLDMF)(1-TLDEF)T\\
\tilde{L}=L-L^{*}\\
P_L(l)=\dfrac{\lambda^{l}e^{-\lambda}}{l!}\\
E(t)=\sum_{i=0}^{\tilde{L}}P_L(i)\cdot t\left(\tilde{L}-i\right)\\
P_L(i)=\dfrac{\lambda^{i}e^{-\lambda}}{i!}\\
t=A_1A_2(1+k)\left(1+\gamma z^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t}\\
z=\left(1-(\chi+\psi+\kappa)\right)\\
\psi=\dfrac{\alpha V}{V+b}\\
\kappa=\beta\dfrac{b}{V+b}\dfrac{W}{W+b_r}\\
W=q_rtr_r+E_r+q_rtr_rE_r\\
A_1=\left(1+e^{-\lim\limits_{a\to\infty}a\tilde{L}}\right)\\
A_2=\left(1+e^{-\lim\limits_{a\to\infty}aV}\right)
\end{cases}
$$

Constraint set

$$
\begin{cases}
c,b,b_r\geq 0,\ n\geq 0.5\\
q,\phi,\delta,\chi,\psi,\kappa,m\in[0;1]\\
\phi+\delta\leq 1\\
\chi+\psi+\kappa\leq 1\\
\beta\leq\alpha\\
E=\begin{cases}E\ \forall q\cdot tr:\ q\cdot tr\in\left[tr^{*};\infty\right)\\ \varnothing\ \forall q\cdot tr:\ q\cdot tr\in\left[0;tr^{*}\right)\end{cases}\\
E_r=\begin{cases}E_r\ \forall q_rtr_r:\ q_rtr_r\in(0;\infty)\\ \varnothing\ \forall q_rtr_r:\ q_rtr_r\notin(0;\infty)\end{cases}
\end{cases}
$$

The model comprises the following set of variables and parameters:

Q	Production capacity measured in units.
N	The number of equipment sets measured in units. Equipment set comprises all machinery and tools defined in the flowchart of the production process.
$E(T)$	Expected operating time of the production unit within a specified period measured in units of time.
$E(t)$	Expected production cycle duration measured in units of time.
da	Total reject rate of production expressed as a fraction of defective articles in total output. Ranges from zero to unity.
$\overline{da_p}$	Reject rate generated by generic labor features expressed as a fraction of defective items in total output. Ranges from zero to unity.
$\overline{da_m}$	Reject rate generated by generic features of machinery expressed as a fraction of defective items in total output. Ranges from zero to unity.
dr	Wear factor of equipment. Measures the rate of equipment deterioration caused by constant use. Ranges from zero to unity.
n	Parameter that determines the pattern of equipment wear factor impact upon the total reject rate. Takes values from zero to infinity.
E	Average direct experience within a worker team measured in units of time.
f	Composite term that determines the speed of improvement in production due to learning in terms of smaller reject rates. It is a function of ability parameters, direct training, and its quality. Lower values of the term correspond to higher speeds of learning. Takes values from zero to unity.

ϕ	Parameter that captures the impact of ability to learn by doing upon the reject rate. Higher values of the term correspond to a greater ability. Ranges from zero to unity.
δ	Composite term that captures the effect of additional training upon the speed of improvement in production due to learning in terms of smaller reject rates. Ranges from zero to unity.
tr	Average amount of direct training within a worker team measured in units of time.
q	Average quality of direct training within a worker team. Accounts for quality of instruction and grades received. Ranges from zero to unity.
tr^*	The minimum amount of direct training that enables the worker to produce measured in time units.
V	Difference between the actual average and minimum required amount of direct training per worker.
η	Parameter that determines the maximum amount of decrease in parameter f that can possibly be produced by direct training. Ranges from zero to unity.
c	Parameter that captures the effect of ability to apply additional knowledge to operations. It determines the speed of convergence of parameter δ to the value η as the difference between the actual average and minimum required amount of direct training per worker increases. Lower values correspond to a greater ability. Takes values from zero to infinity.
da_{setup}	Reject rate generated by equipment setup faults expressed as a fraction of defective items in total output produced by this specific factor. Ranges from zero to unity.
T	Scheduled operating time for the specified period measured in time units.
$TLDMF$	Average operating time loss over machinery failures expressed as a fraction of loss to scheduled operating time. Ranges from zero to unity.
$TLDEF$	Average operating time loss over external factors expressed as a fraction of loss to scheduled operating time.

	External factors include accidents, inspections, etc. Ranges from zero to unity.
L	The number of workers simultaneously employed in production per equipment set.
L^*	The minimum number of workers per equipment set which enables the production to begin. Value is to be taken from the flowchart.
$\tilde{L}$	Difference between the actual and minimum required number of workers per equipment set.
$P_L(l)$	Probability of l out of L workers per equipment set getting sick within a specified period.
λ	Probability of worker being sick within a specified period.
$P_L(i)$	Probability of i out of L workers per equipment set getting sick within a specified period.
t	Production cycle duration measured in time units.
$\overline{t}$	The value production cycle duration takes when inefficiencies of process organization are eliminated and the number of workers and their professional skills are sufficiently high.
d	Parameter that measures potential improvement upon the production cycle duration achievable by increase in the number of workers per equipment set. Expressed in times the amount of $\overline{t}$. Ranges from zero to infinity.
m	Parameter that determines the marginal contribution of additional employee per equipment set.
k	Process organization efficiency. Lower values correspond to higher efficiency. Expressed in times the amount of $\overline{t}$. Ranges from zero to infinity.
γ	Parameter that measures potential improvement upon the production cycle duration achievable by increase in professional skills of employees. Expressed in times the amount of $\overline{t}$. Ranges from zero to infinity.
z	Composite term that determines the speed of improvement in production due to learning in terms of shorter production cycles. It is a function of ability parameters, training, its quality, and related experience. Lower values of the term correspond to higher speeds of learning. Takes values from zero to unity.

χ	Parameter that captures the impact of ability to learn by doing upon the production cycle duration. Higher values of the term correspond to a greater ability. Ranges from zero to unity.
ψ	Composite term that captures the effect of additional direct training upon the speed of improvement in production due to learning in terms of shorter production cycles. Ranges from zero to unity.
κ	Composite term that captures the effect of additional related training and experience upon the speed of improvement in production due to learning in terms of shorter production cycles. Ranges from zero to unity.
α	Parameter that determines the maximum amount of decrease in parameter z that can possibly be produced by direct training. Ranges from zero to unity.
b	Parameter that captures the effect of ability to apply additional knowledge to operations. It determines the speed of convergence of parameter ψ to the value α as the difference between the actual average and minimum required amount of direct training per worker increases. Takes values from zero to infinity.
tr_r	Average amount of related training per worker measured in units of time.
q_r	Average quality of related training within a worker team. Accounts for quality of instruction and grades received. Ranges from zero to unity.
E_r	Average related experience within a worker team measured in units of time.
W	Composite parameter that includes average related training and experience per worker and their interaction term.
β	Parameter that determines the maximum amount of decrease in parameter z that can possibly be produced by parameter W . Ranges from zero to unity.
A_1	Parameter that takes the value of one if difference between the actual and minimum

	required number of workers per equipment set is positive and turns into infinity otherwise.
A_2	Parameter that takes the value of one if difference between the actual average and minimum required amount of direct training per worker is positive and turns into infinity otherwise.

Let us check if this model simultaneously satisfies the entire set of propositions advanced in the previous chapter. In order to do this we have to transform every proposition into an analytical form and then verify whether it is satisfied by the model proposed. Propositions will be verified in the order they were presented.

Proposition 1.1

Statement:

$$t\left(\tilde{L} \prec 0\right) \to \infty$$

Verification:

At first, we take the expression for the production cycle duration that is $t = A_1 A_2 \left(1+k\right)\left(1+\gamma z^E\right)\left(1+dm^{\tilde{L}}\right)\overline{t}$ and set $A_2\left(1+k\right)\left(1+\gamma z^E\right)\overline{t} = \mathrm{R}$ where R is a constant. We can do this since none of the terms included into R contains variable $\tilde{L}$.

$$t = \left| \begin{matrix} \mathrm{R} = A_2\left(1+k\right)\left(1+\gamma z^E\right)\overline{t} \\ A_1 = \left(1+e^{-\lim\limits_{a\to\infty} a\tilde{L}}\right) \end{matrix} \right| = \mathrm{R}\left(1+e^{-\lim\limits_{a\to\infty} a\tilde{L}}\right)\left(1+dm^{\tilde{L}}\right)$$

$$t = \mathrm{R}\left(1+e^{-\lim\limits_{a\to\infty} a\tilde{L}}\right)\left(1+dm^{\tilde{L}}\right) \tag{1}$$

Then we take $\tilde{L} = \tilde{L}_1 \prec 0$ and substitute it into expression (1).

$$t\Big|_{\tilde{L}=\tilde{L}_1} = \mathrm{R}\left(1+e^{-\lim\limits_{a\to\infty} a\tilde{L}_1}\right)\left(1+dm^{\tilde{L}_1}\right) = \left|-\lim_{a\to\infty} a\tilde{L}_1 = +\infty\right| = \mathrm{R}\left(1+e^{+\infty}\right)\left(1+dm^{\tilde{L}_1}\right) =$$
$$= \mathrm{R}\left(1+\infty\right)\left(1+dm^{\tilde{L}_1}\right) = \infty\cdot\mathrm{R}\left(1+dm^{\tilde{L}_1}\right) = \infty \Leftrightarrow t\left(\tilde{L} \prec 0\right) \to \infty$$

Conclusion:

Model completely satisfies Proposition 1.1.

Proposition 1.2

Statement:

$$\begin{cases} \dfrac{\partial t}{\partial \tilde{L}} \le 0 \\ \dfrac{\partial^2 t}{\partial \tilde{L}^2} \ge 0 \end{cases}$$

Verification:

We are interested in positive values of $\tilde{L}$ since according to Proposition 1.1 duration of the production cycle turns into infinity otherwise. Let's work on expression (1) and take the first and the second derivatives of it with respect to $\tilde{L}$.

$$\frac{\partial t}{\partial \tilde{L}} = \left| \begin{array}{c} \lim\limits_{\substack{a \to \infty \\ \tilde{L} \succ 0}} a\tilde{L} = \infty \\ A_1\left(\tilde{L} \succ 0\right) = 1 + e^{-\infty} = \\ = 1 + 0 = 1 \end{array} \right| = \frac{\partial\left(\mathrm{R}\left(1 + dm^{\tilde{L}}\right)\right)}{\partial \tilde{L}} = \underbrace{\ln m}_{\le 0} \underbrace{d\mathrm{R}}_{\ge 0} m^{\tilde{L}} \le 0$$

$$\frac{\partial^2 t}{\partial \tilde{L}^2} = \frac{\partial\left(\ln\left(m\right) d\mathrm{R} m^{\tilde{L}}\right)}{\partial \tilde{L}} = \left(\ln m\right)^2 d\mathrm{R} m^{\tilde{L}}$$

$$\frac{\partial^2 t}{\partial \tilde{L}^2} = \underbrace{\left(\ln m\right)^2 d\mathrm{R}}_{\ge 0} \underbrace{m^{\tilde{L}}}_{\ge 0} \ge 0$$

Conclusion:

Model completely satisfies Proposition 1.2.

Proposition 1.3

Statement:

$$\lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} t = f\left(\overline{t}, \gamma z^{E}\right)$$

Verification:

To verify the statement we take the limit of the expression $t = A_1 A_2 (1+k)\left(1+\gamma z^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t}$ as $\tilde{L}$ goes to infinity and k approaches zero.

$$\lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} t = \lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} A_1 A_2 (1+k)\left(1+\gamma z^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t} = \left| A_1 = 1 + e^{-\lim\limits_{a\to\infty} a\tilde{L}} \right| =$$

$$= \lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} A_2 \left(1+e^{-\lim\limits_{a\to\infty} a\cdot\infty}\right)(1+0)\left(1+\gamma z^{E}\right)\left(1+dm^{\infty}\right)\overline{t} == A_2\left(1+\gamma z^{E}\right)\overline{t} \lim_{\substack{\tilde{L}\to\infty \\ k\to 0}}\left(1+e^{-\infty}\right)(1+d\cdot 0)$$

$$= A_2\left(1+\gamma z^{E}\right)\overline{t} \lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} (1+0)^2 = A_2\left(1+\gamma z^{E}\right)\overline{t} \Rightarrow \lim_{\substack{\tilde{L}\to\infty \\ k\to 0}} t = f\left(\overline{t}, \gamma z^{E}\right)$$

Conclusion:

Model completely satisfies Proposition 1.3.

Proposition 2.1

Statement:

$$\begin{cases} \dfrac{\partial t}{\partial E} \leq 0 \\ \dfrac{\partial^2 t}{\partial E^2} \geq 0 \end{cases}$$

Verification:

At first, we take the expression for the production cycle duration that is $t = A_1 A_2 (1+k)\left(1+\gamma z^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t}$ and set $A_1 A_2 (1+k)\left(1+dm^{\tilde{L}}\right)\overline{t} = H$ where H *is* a constant. We can do this since none of the terms included into H contains variable E.

$$t = \left| A_1 A_2 (1+k)\left(1+dm^{\tilde{L}}\right)\overline{t} = H \geq 0 \right| = H\left(1+\gamma z^{E}\right)$$

$$t = H\left(1+\gamma z^{E}\right) \tag{2}$$

Next we take the first and second derivatives of expression (2) with respect to E.

$$\frac{\partial t}{\partial E} = \frac{\partial\left(H\left(1+\gamma z^{E}\right)\right)}{\partial E} = \underbrace{\gamma H z^{E}}_{\geq 0}\underbrace{\ln z}_{\leq 0} \leq 0$$
$$\frac{\partial^2 t}{\partial E^2} = \frac{\partial\left(\gamma H z^{E}\ln z\right)}{\partial E} = \underbrace{\gamma H z^{E}}_{\geq 0}\underbrace{\ln^2 z}_{\geq 0} \geq 0$$

Conclusion:

Model completely satisfies Proposition 2.1.

Proposition 2.2

Statement:

$$\lim_{E\to 0} t \prec \infty$$

Verification:

To verify the proposition we take the limit of expression (2) as E approaches zero.

$$\lim_{E\to 0} t = \lim_{E\to 0} H\left(1+\gamma z^{E}\right) = H\lim_{E\to 0}\left(1+\gamma z^{0}\right) = H\lim_{E\to 0}\left(1+\gamma\right) = \left(1+\gamma\right)H \prec \infty$$

Conclusion:

Model completely satisfies Proposition 2.2.

Proposition 2.3

Statement:

$$\lim_{E\to\infty} t = f\left(k, \tilde{L}, \overline{t}\right)$$

Verification:

Take the limit of expression (2) as E goes to infinity.

$$\lim_{E\to\infty} t = \lim_{E\to\infty} H\left(1+\gamma z^{E}\right) = H\lim_{E\to\infty}\left(1+\gamma z^{\infty}\right) = H\lim_{E\to\infty}\left(1+\gamma\cdot 0\right) = H\lim_{E\to\infty} 1 = H =$$
$$= \left| H = A_1 A_2 \left(1+k\right)\left(1+dm^{\tilde{L}}\right)\overline{t} \right| = A_1 A_2\left(1+k\right)\left(1+dm^{\tilde{L}}\right)\overline{t} \Rightarrow \lim_{E\to\infty} t = f\left(k,\tilde{L},\overline{t}\right)$$

Conclusion:

Model completely satisfies Proposition 2.3.

Proposition 2.4

Statement:

$$t\left(V \prec 0\right) \to \infty$$

Verification:

Let's take the expression $t = A_1 A_2\left(1+k\right)\left(1+\gamma z^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t}$, expand it with respect to V and combine all terms that do not contain V.

$$t = \left| \begin{array}{l} z = \left(1-\left(\chi+\psi+\kappa\right)\right) \\ \psi = \dfrac{\alpha V}{V+b} \\ \kappa = \beta\dfrac{b}{V+b}\dfrac{W}{W+b_r} \\ A_2 = 1+e^{-\lim\limits_{a\to\infty} aV} \end{array} \right| =$$

$$= A_1\left(1+e^{-\lim\limits_{a\to\infty} aV}\right)\left(1+k\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^{E}\right)\left(1+dm^{\tilde{L}}\right)\overline{t} =$$

$$= \left| A_1\left(1+k\right)\left(1+dm^{\tilde{L}}\right)\overline{t} = J \right| = J\left(1+e^{-\lim\limits_{a\to\infty} aV}\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^{E}\right)$$

$$t = J\left(1+e^{-\lim\limits_{a\to\infty} aV}\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^{E}\right) \qquad (3)$$

Next we substitute $V = V_1 \prec 0$ into the expression (3).

$$t\Big|_{V=V_1\prec 0}=J\left(1+e^{-\lim\limits_{a\to\infty}aV_1}\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V_1}{V_1+b}+\beta\frac{b}{V_1+b}\frac{W}{W+b_r}\right)\right)^E\right)=\left|-\lim_{a\to\infty}aV_1=+\infty\right|=$$

$$=J\left(1+e^{\infty}\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V_1}{V_1+b}+\beta\frac{b}{V_1+b}\frac{W}{W+b_r}\right)\right)^E\right)=$$

$$=J\left(1+\infty\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V_1}{V_1+b}+\beta\frac{b}{V_1+b}\frac{W}{W+b_r}\right)\right)^E\right)=$$

$$=J\cdot\infty\cdot\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V_1}{V_1+b}+\beta\frac{b}{V_1+b}\frac{W}{W+b_r}\right)\right)^E\right)=\infty\Rightarrow t\left(V\prec 0\right)\to\infty$$

Conclusion:

Model completely satisfies Proposition 2.4.

Proposition 2.5

Statement:

$$\begin{cases}\dfrac{\partial t}{\partial V}\le 0\\[2mm] \dfrac{\partial^2 t}{\partial V^2}\ge 0\end{cases}$$

Verification:

We are interested in positive values of V since according to Proposition 2.4 duration of the production cycle turns into infinity otherwise. Let's work on expression (3) and take the first and the second derivatives of it with respect to V.

$$\frac{\partial t}{\partial V}=\left|\begin{matrix}\lim\limits_{\substack{a\to\infty\\ V\succ 0}}aV=\infty\\ A_2\left(V\succ 0\right)=1+e^{-\infty}=\\ =1+0=1\end{matrix}\right|=\frac{\partial\left[J\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right)\right]}{\partial V}=$$

$$=\left| z=1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right| = J\gamma E z^{E-1}\left(-1\right)\left[\alpha\frac{b}{\left(V+b\right)^2}-\beta\frac{b}{\left(V+b\right)^2}\frac{W}{W+b_r}\right]=$$

$$=-J\gamma E z^{E-1}\frac{b}{\left(V+b\right)^2}\left(\alpha-\beta\frac{W}{W+b_r}\right)$$

$$\left.\begin{array}{c}\alpha\geq\beta\\ 0\leq\dfrac{W}{W+b_r}\leq 1\end{array}\right\}\Rightarrow\alpha-\beta\frac{W}{W+b_r}\geq\alpha-\beta\geq 0$$

$$\frac{\partial t}{\partial V}=-\underbrace{J\gamma E z^{E-1}}_{\geq 0}\underbrace{\frac{b}{\left(V+b\right)^2}}_{\geq 0}\underbrace{\left(\alpha-\beta\frac{W}{W+b_r}\right)}_{\geq 0}\leq 0$$

$$\frac{\partial^2 t}{\partial V^2}=\frac{\partial\left(-J\gamma E z^{E-1}\dfrac{b}{\left(V+b\right)^2}\left(\alpha-\beta\dfrac{W}{W+b_r}\right)\right)}{\partial V}=$$

$$=-J\gamma E\left(\alpha-\beta\frac{W}{W+b_r}\right)\left[\left(E-1\right)z^{E-2}\frac{\partial z}{\partial V}\frac{b}{\left(V+b\right)^2}-2\frac{b}{\left(V+b\right)^3}z^{E-1}\right]=$$

$$=\left|\frac{\partial z}{\partial V}=-\frac{b}{\left(V+b\right)^2}\left(\alpha-\beta\frac{W}{W+b_r}\right)\right|=$$

$$=-J\gamma E\left(\alpha-\beta\frac{W}{W+b_r}\right)\left[-\left(E-1\right)z^{E-2}\frac{b}{\left(V+b\right)^2}\left(\alpha-\beta\frac{W}{W+b_r}\right)\frac{b}{\left(V+b\right)^2}-2\frac{b}{\left(V+b\right)^3}z^{E-1}\right]=$$

$$=J\gamma E\left(\alpha-\beta\frac{W}{W+b_r}\right)\left[\left(E-1\right)z^{E-2}\left(\frac{b}{\left(V+b\right)^2}\right)^2\left(\alpha-\beta\frac{W}{W+b_r}\right)+2\frac{b}{\left(V+b\right)^3}z^{E-1}\right]=$$

$$=J\gamma E z^{E-1}\frac{b}{\left(V+b\right)^2}\left(\alpha-\beta\frac{W}{W+b_r}\right)\frac{1}{V+b}\left[\left(E-1\right)z^{-1}\frac{b}{V+b}\left(\alpha-\beta\frac{W}{W+b_r}\right)+2\right]=$$

$$=\left(-\frac{\partial t}{\partial V}\right)\frac{1}{V+b}\left[\left(E-1\right)z^{-1}\frac{b}{V+b}\left(\alpha-\beta\frac{W}{W+b_r}\right)+2\right]$$

$$E \geq 1: \frac{\partial^2 t}{\partial V^2} = \underbrace{\left(-\frac{\partial t}{\partial V}\right)\frac{1}{V+b}}_{\geq 0} \underbrace{\left[(E-1)z^{-1}\frac{b}{V+b}\left(\alpha - \beta\frac{W}{W+b_r}\right)+2\right]}_{\geq 0} \geq 0$$

For $0 \leq E \leq 1$ the sign of the second derivative of t with respect to V depends on the magnitudes of other factors.

Conclusion:

Model partially satisfies Proposition 2.5.

Proposition 2.6

Statement:

$$\lim_{V\to\infty} t = f\left(\bar{t}, k, \tilde{L}, E, ...\right)$$

Verification:

To verify the proposition we take the limit of expression (3) as V tends to infinity.

$$\lim_{V\to\infty} t = \lim_{V\to\infty} J\left(1+e^{-\lim_{a\to\infty} aV}\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right) =$$

$$= J\left(1+e^{-\lim_{\substack{a\to\infty\\V\to\infty}} aV}\right)\left(1+\gamma\left(1-\left(\chi+\lim_{V\to\infty}\frac{\alpha V}{V+b}+\beta\lim_{V\to\infty}\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right) =$$

$$= \left| \begin{array}{c} \lim_{\substack{a\to\infty\\V\to\infty}} aV = \infty \\ \lim_{V\to\infty}\frac{\alpha V}{V+b} = \lim_{V\to\infty}\frac{(\alpha V)'}{(V+b)'} = \lim_{V\to\infty}\frac{\alpha}{1} = \alpha \\ \lim_{V\to\infty}\frac{b}{V+b} = 0 \end{array} \right| = J\left(1+e^{-\infty}\right)\left(1+\gamma\left(1-(\chi+\alpha)\right)^E\right) = J\left(1+\gamma\left(1-(\chi+\alpha)\right)^E\right) =$$

$$= \left| J = A_1(1+k)\left(1+dm^{\tilde{L}}\right)\bar{t} \right| = A_1(1+k)\left(1+\gamma\left(1-(\chi+\alpha)\right)^E\right)\left(1+dm^{\tilde{L}}\right)\bar{t} \Rightarrow$$

$$\Rightarrow \lim_{V\to\infty} t = f\left(\overline{t}, k, \tilde{L}, E, ...\right)$$

Conclusion:

Model completely satisfies Proposition 2.6.

Proposition 2.7

Statement:

$$\left.\frac{\partial t}{\partial tr}\right|_{q=0} = 0$$

Verification:

We are interested in positive values of V since according to Proposition 2.4 duration of the production cycle turns into infinity otherwise. Let's work on expression (3) and take the first derivative of it with respect to *tr*.

$$\frac{\partial t}{\partial tr} = \left| \begin{matrix} \lim\limits_{\substack{a\to\infty \\ V\succ 0}} aV = \infty \\ A_2\left(V \succ 0\right) = 1 + e^{-\infty} = \\ = 1 + 0 = 1 \end{matrix} \right| = \frac{\partial\left[J\left(1+\gamma\left(1-\left(\chi + \frac{\alpha V}{V+b} + \beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right)\right]}{\partial tr} =$$

$$= \left| z = 1 - \left(\chi + \frac{\alpha V}{V+b} + \beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right| = -J\gamma E z^{E-1}\left[\alpha q\frac{b}{\left(V+b\right)^2} - \beta q\frac{b}{\left(V+b\right)^2}\frac{W}{W+b_r}\right] =$$

$$= -qJ\gamma E z^{E-1}\frac{b}{\left(V+b\right)^2}\left[\alpha - \beta\frac{W}{W+b_r}\right]$$

Now we substitute zero value of q into the expression for the first derivative of t with respect to *tr*.

$$\left.\frac{\partial t}{\partial tr}\right|_{q=0} = \left| \begin{matrix} V\big|_{q=0} = 0\cdot tr - tr^* = \\ = -tr^* \end{matrix} \right| =$$

$$= -0\cdot J\gamma E\left(1-\left(\chi + \frac{-\alpha tr^*}{-tr^*+b} + \beta\frac{b}{-tr^*+b}\frac{W}{W+b_r}\right)\right)^{E-1}\frac{b}{\left(-tr^*+b\right)^2}\left[\alpha - \beta\frac{W}{W+b_r}\right] = 0$$

Conclusion:

Model completely satisfies Proposition 2.7.

Proposition 2.8

Statement:

$$\frac{\partial \kappa}{\partial V} \le 0$$

Verification:

Let's take the expression for *k* and differentiate it with respect to *V*.

$$\frac{\partial \kappa}{\partial V} = \frac{\partial\left(\beta \frac{b}{V+b}\frac{W}{W+b_r}\right)}{\partial V} = -\beta \frac{b}{(V+b)^2}\frac{W}{W+b_r}$$

$$\frac{\partial \kappa}{\partial V} = -\underbrace{\beta}_{\ge 0} \frac{b}{\underbrace{(V+b)^2}_{\ge 0}}\frac{W}{\underbrace{W+b_r}_{\ge 0}} \le 0$$

Conclusion:

Model completely satisfies Proposition 2.8.

Proposition 2.9

Statement:

$$\begin{cases} \frac{\partial t}{\partial W} \le 0 \\ \frac{\partial^2 t}{\partial W^2} \ge 0 \end{cases}$$

Verification:

To prove this statement we take the first and the second derivatives of expression (3) with respect to *W*.

$$\frac{\partial t}{\partial W}=\left|A_2=1+e^{-\lim_{a\to\infty}aV}\right|=\frac{\partial\left(JA_2\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right)\right)}{\partial W}=$$

$$=\left|z=1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right|=JA_2\gamma Ez^{E-1}(-1)\beta\frac{b}{V+b}\frac{b_r}{\left(W+b_r\right)^2}=$$

$$=-\underbrace{JA_2\gamma Ez^{E-1}\beta}_{\geq 0}\underbrace{\frac{b}{V+b}}_{\geq 0}\underbrace{\frac{b_r}{\left(W+b_r\right)^2}}_{\geq 0}\leq 0\Leftrightarrow\frac{\partial t}{\partial W}\leq 0$$

$$\frac{\partial^2 t}{\partial W^2}=\frac{\partial\left(-JA_2\gamma Ez^{E-1}\beta\frac{b}{V+b}\frac{b_r}{\left(W+b_r\right)^2}\right)}{\partial W^2}=$$

$$=-JA_2\gamma E\beta\frac{b}{V+b}\left[(E-1)z^{E-2}\frac{\partial z}{\partial W}\frac{b_r}{\left(W+b_r\right)^2}-2z^{E-1}\frac{b_r}{\left(W+b_r\right)^3}\right]=$$

$$=\left|\frac{\partial z}{\partial W}=-\beta\frac{b}{V+b}\frac{b_r}{\left(W+b_r\right)^2}\right|=-JA_2\gamma E\beta\frac{b}{V+b}\left[-(E-1)z^{E-2}\beta\frac{b}{V+b}\left(\frac{b_r}{\left(W+b_r\right)^2}\right)^2-\right.$$

$$\left.-2z^{E-1}\frac{b_r}{\left(W+b_r\right)^3}\right]=JA_2\gamma Ez^{E-1}\beta\frac{b}{V+b}\frac{b_r}{\left(W+b_r\right)^3}\left[(E-1)z^{-1}\beta\frac{b}{V+b}\frac{b_r}{W+b_r}+2\right]=$$

$$=\left|\left(-\frac{\partial t}{\partial W}\right)=JA_2\gamma Ez^{E-1}\beta\frac{b}{V+b}\frac{b_r}{\left(W+b_r\right)^2}\right|=\left(-\frac{\partial t}{\partial W}\right)\frac{1}{W+b_r}\left[(E-1)z^{-1}\beta\frac{b}{V+b}\frac{b_r}{W+b_r}+2\right]$$

$$E\geq 1:\frac{\partial^2 t}{\partial W^2}=\underbrace{\left(-\frac{\partial t}{\partial W}\right)\frac{1}{W+b_r}}_{\geq 0}\underbrace{\left[(E-1)z^{-1}\beta\frac{b}{V+b}\frac{b_r}{W+b_r}+2\right]}_{\geq 0}\geq 0$$

For $0\leq E\leq 1$ the sign of the second derivative of *t* with respect to *W* depends on the magnitudes of other factors.

Conclusion:

Model partially satisfies Proposition 2.9.

Proposition 2.10

Statement:

$$\lim_{W\to\infty} t = f\left(\overline{t}, k, \tilde{L}, E, V, \ldots\right)$$

Verification:

To verify the proposition we take the limit of expression (3) as W tends to infinity.

$$\lim_{W\to\infty} t = \left| A_2 = 1 + e^{-\lim_{a\to\infty} aV} \right| = \lim_{W\to\infty} JA_2\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\frac{W}{W+b_r}\right)\right)^E\right) =$$

$$= JA_2\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\lim_{W\to\infty}\frac{W}{W+b_r}\right)\right)^E\right) = \left| \begin{array}{c} \lim_{W\to\infty}\frac{W}{W+b_r} = \lim_{W\to\infty}\frac{W'}{\left(W+b_r\right)'} = \\ = \lim_{W\to\infty}\frac{1}{1} = 1 \end{array} \right| =$$

$$= JA_2\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\right)\right)^E\right) = \left| J = A_1\left(1+k\right)\left(1+dm^{\tilde{L}}\right)\overline{t} \right| =$$

$$= A_1A_2\left(1+k\right)\left(1+\gamma\left(1-\left(\chi+\frac{\alpha V}{V+b}+\beta\frac{b}{V+b}\right)\right)^E\right)\left(1+dm^{\tilde{L}}\right)\overline{t} \Rightarrow \lim_{W\to\infty} t = f\left(\overline{t}, k, \tilde{L}, E, V, \ldots\right)$$

Conclusion:

Model completely satisfies Proposition 2.10.

Proposition 3.1

Statement:

$$E\left(t\right) = P\left(\lambda\right)_1 \cdot t\left(\tilde{L}_1\right) + P\left(\lambda\right)_2 \cdot t\left(\tilde{L}_2\right) + \ldots + P\left(\lambda\right)_n \cdot t\left(\tilde{L}_n\right)$$

Verification:

To verify the statement we take the expression for the expected production cycle duration and expand it.

$$E(t)=\sum_{i=0}^{\tilde{L}} P_L(i)\cdot t(\tilde{L}-i)=P_L(0)\cdot t(\tilde{L})+P_L(1)\cdot t(\tilde{L}-1)+...+P_L(\tilde{L})\cdot t(0)=\left|P_L(i)=\frac{\lambda^i e^{-\lambda}}{i!}\right|=$$

$$=e^{-\lambda}t(\tilde{L})+\lambda e^{-\lambda}t(\tilde{L}-1)+...+\frac{\lambda^{\tilde{L}}e^{-\lambda}}{\tilde{L}!}t(0)=\left|\begin{array}{c} e^{-\lambda}=P(\lambda)_1 \\ \lambda e^{-\lambda}=P(\lambda)_2 \\ \frac{\lambda^{\tilde{L}}e^{-\lambda}}{\tilde{L}!}=P(\lambda)_n \\ \tilde{L}=\tilde{L}_1 \\ \tilde{L}-1=\tilde{L}_2 \\ 0=\tilde{L}_n \end{array}\right|=$$

$$=P(\lambda)_1\cdot t(\tilde{L}_1)+P(\lambda)_2\cdot t(\tilde{L}_2)+...+P(\lambda)_n\cdot t(\tilde{L}_n)$$

Conclusion:

Model completely satisfies Proposition 3.1.

Proposition 3.2

Statement:

$$E(T)\prec T\ \forall\lambda:\lambda\succ 0$$

Verification:

To verify the statement we take $\lambda=\tilde{\lambda}\succ 0$ and substitute it into the expression for expected operating time

$$E(T)=\left[1-\sum_{l=\tilde{L}}^{L} P_L(l)\right](1-TLDMF)(1-TLDEF)T \tag{4}$$

$$E(T)\Big|_{\substack{TLDMF=0\\TLDEF=0\\ \lambda=\tilde{\lambda}}}=\left[1-\sum_{l=\tilde{L}}^{L} P_L(l)\right]T=\left|P_L(l)=\frac{\tilde{\lambda}^l e^{-\tilde{\lambda}}}{l!}\right|=\left[1-\sum_{l=\tilde{L}}^{L}\frac{\tilde{\lambda}^l e^{-\tilde{\lambda}}}{l!}\right]T=$$

$$=\left[1-\underbrace{\left(\frac{\tilde{\lambda}^{\tilde{L}}e^{-\tilde{\lambda}}}{\tilde{L}!}+\frac{\tilde{\lambda}^{(\tilde{L}+1)}e^{-\tilde{\lambda}}}{(\tilde{L}+1)!}+\ldots+\frac{\tilde{\lambda}^{L}e^{-\tilde{\lambda}}}{L!}\right)}_{\succ 0}\right]T\prec T$$

Conclusion:

Model completely satisfies Proposition 3.2.

Proposition 4.1

Statement:

$$E(T)\leq T$$

Verification:

We will work on the expression (4) and its components.

$$E(T)=\left|\begin{array}{l}\left[1-\sum_{l=\tilde{L}}^{L}P_L(l)\right]=a_1\\(1-TLDMF)=a_2\\(1-TLDEF)=a_3\end{array}\right|=\prod_{j=1}^{3}a_jT$$

$$a_1,a_2,a_3\in[0;1]\Rightarrow\prod_{j=1}^{3}a_j\in[0;1]\Rightarrow\prod_{j=1}^{3}a_jT\leq T\Leftrightarrow E(T)\leq T$$

Conclusion:

Model completely satisfies Proposition 4.1.

Proposition 5.1

Statement:

$$da=1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)$$

Verification:

Let $\tilde{Q}_0$ denote the outcome when the item produced has no defects

$\tilde{Q}_1$ - outcome when the item produced has the defects generated by generic labor features

$\tilde{Q}_2$ - outcome when the item produced has the defects generated by generic features of machinery

$\tilde{Q}_3$ - outcome when the item produced has the defects generated by equipment deterioration

$\tilde{Q}_4$ - outcome when the item produced has the defects generated by insufficient experience of employees

$\tilde{Q}_5$ - outcome when the item produced has the defects generated by equipment setup faults.

Then the probability of producing the specified item equals

$$P\left(\tilde{Q}_0\right) = 1 - P\left(\tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3 \cup \tilde{Q}_4 \cup \tilde{Q}_5\right)$$

According to the additional rule of probabilities

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When A and B are statistically independent

$$P(A \cap B) = P(A / B) \cdot P(B) = P(B / A) \cdot P(A) = P(A) \cdot P(B)$$

Substitution into the expression for $P(A \cup B)$ yields

$$P(A \cup B) = P(A) + P(B) - P\left(A\right)P\left(B\right)$$

If we subtract both sides of the latter expression from unity we obtain

$$1 - P(A \cup B) = 1 - P(A) - P(B) + P\left(A\right)P\left(B\right) = \left(1 - P(B)\right) - P\left(A\right)\left(1 - P(B)\right) = \left(1 - P\left(A\right)\right)\left(1 - P(B)\right)$$

Since the factors that generate defects are independent, collectively exhaustive and nonexclusive, application to our case yields

$$P\left(\tilde{Q}_0\right)=\prod_{j=1}^{5}\left(1-P\left(\tilde{Q}_1\right)\right)=\left|\begin{matrix}P\left(\tilde{Q}_1\right)=\overline{da_p}\\ P\left(\tilde{Q}_2\right)=\overline{da_m}\\ P\left(\tilde{Q}_3\right)=dr^{2n}\\ P\left(\tilde{Q}_4\right)=f^{E}\\ P\left(\tilde{Q}_5\right)=da_{setup}\end{matrix}\right|=\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)$$

Then the probability of producing the defective item equals

$$P\left(\tilde{Q}_1\cup\tilde{Q}_2\cup\tilde{Q}_3\cup\tilde{Q}_4\cup\tilde{Q}_5\right)=1-P\left(\tilde{Q}_0\right)=$$
$$=1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)=da$$

Conclusion:

Model completely satisfies Proposition 5.1.

Proposition 5.2

Statement:

$$\lim_{\substack{dr\to 0\\ f^{E}\to 0\\ da_{setup}\to 0}} da = g\left(\overline{da_p},\overline{da_m}\right)$$

Verification:

Let's take the limit of the expression for *da* as equipment wear factor, setup and professional skills term simultaneously approach zero.

$$\lim_{\substack{dr\to 0\\ f^{E}\to 0\\ da_{setup}\to 0}} da = \lim_{\substack{dr\to 0\\ f^{E}\to 0\\ da_{setup}\to 0}} \left(1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)\right) = \lim_{\substack{dr\to 0\\ f^{E}\to 0\\ da_{setup}\to 0}} \left(1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\right) =$$

$$= 1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right) \Rightarrow \lim_{\substack{dr\to 0\\ f^{E}\to 0\\ da_{setup}\to 0}} da = g\left(\overline{da_p}, \overline{da_m}\right)$$

Conclusion:

Model completely satisfies Proposition 5.2.

Proposition 5.3

Statement:

$$\left(\overline{da_p}\to 1\right)\cup\left(\overline{da_m}\to 1\right)\cup\left(dr^{2n}\to 1\right)\cup\left(f^{E}\to 1\right)\cup\left(da_{setup}\to 1\right)\Rightarrow Q=0$$

Verification:

Let us take the expression for production capacity and perform a series of simple substitutions.

$$Q=\left(1-da\right)N\cdot\frac{E(T)}{E(t)} = \left| \begin{array}{c} da = 1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right) \\ I = N\cdot\dfrac{E(T)}{E(t)} \end{array} \right| =$$

$$= \left[1-1+\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right)\right] I =$$

$$= \left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^{E}\right)\left(1-da_{setup}\right) I$$

$$Q = \begin{vmatrix} \overline{da_p} = s_1 \\ \overline{da_m} = s_2 \\ dr^{2n} = s_3 \\ f^{E} = s_4 \\ da_{setup} = s_5 \end{vmatrix} = \prod_{i=1}^{5} (1 - s_i) I$$

$$\lim_{s_j \to 1} Q = \lim_{s_j \to 1} \left[(1 - s_j) \prod_{i=1}^{j-1} (1 - s_i) \prod_{l=j+1}^{5} (1 - s_l) I \right] = \prod_{i=1}^{j-1} (1 - s_i) \prod_{l=j+1}^{5} (1 - s_l) I \lim_{s_j \to 1} (1 - s_j) =$$

$$= \prod_{i=1}^{j-1} (1 - s_i) \prod_{l=j+1}^{5} (1 - s_l) I \cdot 0 = 0$$

Conclusion:

Model completely satisfies Proposition 5.3.

Proposition 5.4

Statement:

$$\lim_{E \to 0} da = 1$$

Verification:

Let's take the limit of expression for *da* as direct experience approaches zero.

$$\lim_{E \to 0} da = \lim_{E \to 0} \left[1 - \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - dr^{2n}\right)\left(1 - f^{E}\right)\left(1 - da_{setup}\right) \right] =$$

$$= 1 - \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - dr^{2n}\right)\left(1 - da_{setup}\right) \lim_{E \to 0} \left(1 - f^{E}\right) =$$

$$= 1 - \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - dr^{2n}\right)\left(1 - da_{setup}\right) \lim_{E \to 0} \left(1 - f^{0}\right) =$$

$$= 1 - \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - dr^{2n}\right)\left(1 - da_{setup}\right)(1 - 1) =$$

$$= 1 - \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - dr^{2n}\right)\left(1 - da_{setup}\right) \cdot 0 = 1 - 0 = 1$$

Conclusion:

Model completely satisfies Proposition 5.4.

Proposition 5.5

Statement:

$$\begin{cases} \dfrac{\partial da}{\partial E} \leq 0 \\ \dfrac{\partial^2 da}{\partial E^2} \geq 0 \end{cases}$$

Verification:

Let's take the expression for *da* and differentiate it twice with respect to direct experience.

$$\frac{\partial da}{\partial E} = \frac{\partial \left[1-\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-f^E\right)\left(1-da_{setup}\right)\right]}{\partial E} =$$

$$= \left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)\frac{\partial\left(1-f^E\right)}{\partial E} = \left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)f^E \ln f$$

$$\frac{\partial da}{\partial E} = \underbrace{\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)f^E}_{\geq 0}\underbrace{\ln f}_{\leq 0} \leq 0$$

$$\frac{\partial^2 da}{\partial E^2} = \frac{\partial\left[\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)f^E \ln f\right]}{\partial E} =$$

$$= \left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)f^E\left(\ln f\right)^2$$

$$\frac{\partial^2 da}{\partial E^2} = \underbrace{\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right)f^E}_{\geq 0}\underbrace{\left(\ln f\right)^2}_{\geq 0} \geq 0$$

Conclusion:

Model completely satisfies Proposition 5.5.

Proposition 5.6

Statement:

$$\begin{cases} \dfrac{\partial da}{\partial V} \le 0 \\ \dfrac{\partial^2 da}{\partial V^2} \ge 0 \end{cases}$$

Verification:

Let's take the expression for *da* and perform a simple substitution.

$$da = \left|\left(1-\overline{da_p}\right)\left(1-\overline{da_m}\right)\left(1-dr^{2n}\right)\left(1-da_{setup}\right) = P\right| = 1-P\left(1-f^E\right)$$

$$da = 1-P\left(1-f^E\right) \tag{5}$$

Then we differentiate expression (5) twice with respect to V.

$$\frac{\partial da}{\partial V} = \frac{\partial\left(1-P\left(1-f^E\right)\right)}{\partial V} = PEf^{E-1}\frac{\partial f}{\partial V} = \left| \begin{array}{l} \dfrac{\partial f}{\partial V} = -\dfrac{\partial \delta}{\partial V} \\ \dfrac{\partial \delta}{\partial V} = \dfrac{\eta c}{\left(V+c\right)^2} \end{array} \right| = -PEf^{E-1}\frac{\eta c}{\left(V+c\right)^2}$$

$$\frac{\partial da}{\partial V} = -\underbrace{PEf^{E-1}}_{\ge 0}\underbrace{\frac{\eta c}{\left(V+c\right)^2}}_{\ge 0} \le 0$$

$$\frac{\partial^2 da}{\partial V^2} = \frac{\partial\left(-PEf^{E-1}\dfrac{\eta c}{\left(V+c\right)^2}\right)}{\partial V} = -PE\eta c\left[\left(E-1\right)f^{E-2}\left(-\frac{\eta c}{\left(V+c\right)^2}\right)\frac{1}{\left(V+c\right)^2} - 2f^{E-1}\frac{1}{\left(V+c\right)^3}\right] =$$

$$= PEf^{E-1}\frac{\eta c}{\left(V+c\right)^2}\frac{1}{\left(V+c\right)}\left[\left(E-1\right)f^{-1}\frac{\eta c}{\left(V+c\right)}+2\right] = \left|PEf^{E-1}\frac{\eta c}{\left(V+c\right)^2} = -\frac{\partial da}{\partial V}\right| =$$

$$= \left(-\frac{\partial da}{\partial V}\right)\frac{1}{\left(V+c\right)}\left[\left(E-1\right)f^{-1}\frac{\eta c}{\left(V+c\right)}+2\right]$$

$$E \ge 1: \frac{\partial^2 da}{\partial V^2} = \underbrace{\left(-\frac{\partial da}{\partial V}\right)}_{\ge 0}\underbrace{\frac{1}{\left(V+c\right)}}_{\ge 0}\underbrace{\left[\left(E-1\right)f^{-1}\frac{\eta c}{\left(V+c\right)}+2\right]}_{\ge 0} \ge 0$$

For $0 \le E \le 1$ the sign of the second derivative of *da* with respect to V depends on the magnitudes of other factors.

Conclusion:

Model partially satisfies Proposition 5.6.

Proposition 5.7

Statement:

$$\lim_{V \to \infty} da = const \ge 0$$

Verification:

To verify the statement we take the limit of expression (5) as V tends to infinity.

$$\lim_{V \to \infty} da = \lim_{V \to \infty} \left(1 - P\left(1 - f^{E}\right)\right) = \left| \begin{array}{c} f = \left(1 - (\phi + \delta)\right) \\ \delta = \dfrac{\eta V}{V + c} \end{array} \right| = \lim_{V \to \infty} \left(1 - P\left(1 - \left(1 - \left(\phi + \frac{\eta V}{V + c}\right)\right)^{E}\right)\right) =$$

$$= 1 - P\left(1 - \left(1 - \left(\phi + \lim_{V \to \infty} \frac{\eta V}{V + c}\right)\right)^{E}\right) = \left| \begin{array}{c} \lim_{V \to \infty} \dfrac{\eta V}{V + c} = \lim_{V \to \infty} \dfrac{(\eta V)'}{(V + c)'} = \\ = \lim_{V \to \infty} \dfrac{\eta}{1} = \eta \end{array} \right| = 1 - P\left(1 - \left(1 - (\phi + \eta)\right)^{E}\right)$$

$$\phi + \delta \le 1 \Rightarrow \phi + \eta \le 1 \Rightarrow 1 - (\phi + \eta) \le 1 \Rightarrow 1 - \left(1 - (\phi + \eta)\right) \le 1 \Rightarrow \left(1 - \left(1 - (\phi + \eta)\right)\right)^{E} \le 1$$

$$\left. \begin{array}{c} \left(1 - \left(1 - (\phi + \eta)\right)\right)^{E} \le 1 \\ P \le 1 \end{array} \right\} \Rightarrow 1 - P\left(1 - \left(1 - (\phi + \eta)\right)^{E}\right) = \lim_{V \to \infty} da \ge 0$$

Conclusion:

Model completely satisfies Proposition 5.7.

Proposition 5.8

Statement:

$$\begin{cases} \dfrac{\partial da}{\partial dr} \geq 0 \\ \dfrac{\partial^2 da}{\partial dr^2} \geq 0 \end{cases}$$

Verification:

Let's take the expression for *da* and perform a simple substitution.

$$da = \left| \left(1 - \overline{da_p}\right)\left(1 - \overline{da_m}\right)\left(1 - f^E\right)\left(1 - da_{setup}\right) = U \right| = 1 - U\left(1 - dr^{2n}\right)$$

$$da = 1 - U\left(1 - dr^{2n}\right) \tag{6}$$

Next we differentiate the expression (6) twice with respect to *dr*.

$$\frac{\partial da}{\partial dr} = \frac{\partial\left(1 - U\left(1 - dr^{2n}\right)\right)}{\partial dr} = 2Undr^{2n-1}$$

$$\frac{\partial da}{\partial dr} = 2\underbrace{Un}_{\geq 0}\underbrace{dr^{2n-1}}_{\geq 0} \geq 0$$

$$\frac{\partial^2 da}{\partial dr^2} = \frac{\partial\left(2Undr^{2n-1}\right)}{\partial dr} = 2Un\left(2n-1\right)dr^{2(n-1)}$$

$$\frac{\partial^2 da}{\partial dr^2} = 2\underbrace{Un\left(2n-1\right)}_{\geq 0}\underbrace{dr^{2(n-1)}}_{\geq 0} \geq 0$$

Conclusion:

Model completely satisfies Proposition 5.8.

Thus we end up with majority of the propositions being completely satisfied by the specified function. The rest are satisfied only partially. However, this drawback of the framework is rather minor since it involves relatively short intervals of the admitted regions of the variables.

Since the model has been proved analytically, we can discuss the issue of optional engineering terms. It is obvious that some blocks of the model constitute an approximation of real production practices. It is especially true for the block describing the behavior of the production capacity function with respect to the process organization. Let's recall the structure of the production cycle duration term:

$$t = A_1 A_2 \boxed{(1+k)} \left(1+\gamma z^E\right)\left(1+dm^L\right)\overline{t}$$

Within this term the entire process organization efficiency is squeezed into a single parameter that may introduce considerable bias in terms of prediction. Therefore, considering the improvement of the model performance one may resort to the complex process organization models that lay in the field of production management. It is desirable that those models account for the separate effects of job design, spatial arrangement, time profile, and production structure. Even though resulting in the considerable complication of the framework, the introduction of the engineering terms will certainly boost the forecasting power of the framework since it increases the precision of the modeling.

CONCLUSION

The issue of production capacity raised in the thesis has not received much attention from economists. Instead, they focused primarily upon the concept of production function that was distantly related to production capacity. Still in the second half of the twentieth century a number of attempts were made to combine the findings of production engineering and microeconomics. Those researchers managed to discern the potential benefits of this approach. However, complexity of the issue impeded further development and research did not go beyond uncoordinated efforts. At the same time, the progress of production management was more substantial. Engineers managed to single out and classify the majority of the elements that comprise the production system and determine the production capacity of the enterprise. A considerable number of studies were conducted to measure the impact of various factors of production upon capacity. Yet the results remain scattered and this puts severe limits upon the applicability of separate findings.

In this paper we formulate the production capacity function, designed for the application in the enterprise management. Its main component is a basic identity between the quantity of output and operating time, which is expanded to account for the range of factors that make a significant impact upon the capacity. In order to verify the framework consistency, a set of propositions, which are argued to correctly describe the essential features of any production in manufacturing, is advanced. This set comprises the propositions on labor, process organization, professional skills of employees, operating time, and reject rate of production.

After the presentation of the framework, we analytically verify whether it satisfies the propositions advanced. At first, each proposition is analytically interpreted in order for us to be able to apply the math apparatus. Next, we perform the required operations to check whether each separate proposition is satisfied. As a result, we find out that the model simultaneously satisfies the entire set of propositions. However, some of them are satisfied only partially. Yet this drawback is rather minor since it involves relatively short intervals of the admitted regions of the variables.

There is also a possibility to improve the framework by inclusion of the composite engineering term instead of a single parameter for the process organization efficiency.

To sum up, thesis introduces the applicable framework for the quantitative analysis of production capacity of enterprises.

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